Limit sets of stable Cellular Automata

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Abstract. We study limit sets of stable cellular automata standing from a symbolic dynamics point of view where they are a special case of sofic shifts admitting a steady epimorphism. We prove that there exists a right-closing almost-everywhere steady factor map from one irreducible sofic shift onto another one if and only if there exists such a map from the domain onto the minimal right-resolving cover of the image. We define right-continuing almost-everywhere steady maps and prove that there exists such a steady map between two sofic shifts if and only if there exists a factor map from the domain onto the minimal right-resolving cover of the image.

In terms of cellular automata, this translates into: A sofic shift can be the limit set of a stable cellular automaton with a right-closing almost-everywhere dynamics onto its limit set if and only if it is the factor of a fullshift and there exists a right-closing almost-everywhere factor map from the sofic shift onto its minimal right-resolving cover. A sofic shift can be the limit set of a stable cellular automaton reaching its limit set with a right-continuing almost-everywhere factor map if and only if it is the factor of a fullshift and there exists a factor map from the sofic shift onto its minimal right-resolving cover.

Finally, as a consequence of the previous results, we provide a characterization of the Almost of Finite Type shifts (AFT) in terms of a property of steady maps that have them as range.

Cellular automata were introduced by Von Neumann as a model of some biological processes [**17**] and have become a rich model of complex systems: systems with simple local behavior but complex global evolution. Many different points of view have been adopted to formalize this complexity, using methods of combinatorics, topology, ergodic theory, language theory and theory of

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computation. The first, and maybe most known, systematic study of this complex behavior was performed by S. Wolfram [**21**] by doing computer experiments and then analyzing the observed behavior of the cellular automaton. From a mathematical point of view, the long-term behavior of a cellular automaton can be modelled by its dynamics on its limit set: The set of configurations that can be reached arbitrary late in the evolution of the automaton. We can distinguish between two types of cellular automata by their ways of reaching their limit set, starting from the fullshift, the set of all possible configurations [**15**]: Either the automaton reaches its limit set in finite time, the cellular automaton is then called *stable*, or it never reaches it and only gets closer and closer to it, the cellular automaton is then called *unstable*.

In this paper, we are interested in the, *a priori*, simpler case of stable cellular automata for which it is still an important open problem to obtain a characterization of their limit sets [**7**, Section 16]. Stable cellular automata can be modelled in terms of symbolic dynamics [**14, 15**]: They are a special case of the steady factor maps of Barth and Dykstra [**4**]. Basic remarks yield necessary conditions for a subshift to be the limit set of a stable cellular automaton: This is what A. Maass called property (H) [**15**]. A. Maass then proved that these necessary conditions are also sufficient for a large class of sofic shifts: The almost of finite type (AFT [**16**]) shifts [**15**, Theorem 4.8]. Albeit not exactly stated as such in A. Maass' paper, his methods for constructing limit sets of stable cellular automata are to obtain a weak conjugacy between two subshifts (constructing factor maps from each subshift onto the other one) and then if one can prove that one subshift is a stable limit set of cellular automata then the other one is automatically also a stable limit set [**3**, Lemma 4.1]. As a consequence of Boyle's extension lemma [**5**, Lemma 2.4], a subshift of finite type (SFT) having property (H) is the stable limit set of a cellular automaton [**15**, Theorem 3.2]. Moreover, all the methods we know for constructing stable cellular automata go through a weak conjugacy with an SFT, this is what led us to the following conjecture that we restate here:

Conjecture 1. *[**3**, Conjecture 1] The limit set of any stable cellular automaton is weakly conjugate to an SFT.*

After fixing the definitions in Section 1, where we adopt the vocabulary from symbolic dynamics [**14**], we prove the basic results that we will use along the rest of the paper. In Section 2, we prove that a sofic shift is the stable limit set of a cellular automaton with a right-closing almost-everywhere dynamics on its limit set if and only if the sofic shift has property (H) and there exists a right-closing almost-everywhere factor map from the sofic shift onto its minimal right-resolving cover. In Section 3, we prove that a cellular automata attains its limit set by a right-continuing almost-everywhere factor map if and only if its limit set factors onto its minimal right-resolving cover. By similar methods, we provide in Section 4 a characterization of the almost of finite type (AFT) shifts of B. Marcus [**16**] in terms of the range of a special class of steady maps and characterize AFT stable limit sets of cellular automata as those that can be attained by a left and right-continuing

almost-everywhere cellular automaton.

Each of these three sections (2, 3 and 4) are organized in the same way and each of them provides a characterization in terms of steady maps (Theorems 2.2, 3.2 and 4.2 respectively). One direction of each of these characterizations always makes use of an extension theorem for sliding block codes: these are, respectively, Boyle's extension lemma [**5**, Lemma 2.4], its refinement by Boyle and Tuncel [**10**, Theorem 5.3] and yet another refinement by Jung [**13**, Theorem 4.5]. Hence, Sections 2, 3 and 4 are organized in a somewhat chronological order of the results they are based on.

1 Definitions and basic results

Let A be a finite set, called the *alphabet* embedded with the discrete topology. Consider $A^{\mathbb{Z}}$ as the *fullshift* over A embedded with the product topology. For $i \in \mathbb{Z}$ and $x \in A^{\mathbb{Z}}$, denote by x_i the value of x at position i. A metric for the topology of $A^{\mathbb{Z}}$ can be defined for example as $d(x,y) = 2^{-\min\{|i|, x_i \neq y_i\}}$.

Words and languages A *word* over A is an element of $A^* = \cup_{n\in\mathbb{N}} A^n$. Denote by $|w|$ the *length* of the word w, *i.e.*, such that $w \in A^{|w|}$. For $i < j \in \mathbb{Z}$ and $x \in A^{\mathbb{Z}}$, denote by $x_{[i;j]}$ the word $x_i x_{i+1} \dots x_j \in A^*$. We say that a word w *appears* in $x \in A^{\mathbb{Z}}$ at position i if $x_{[i;i+|w|-1]} = w$. For a subset $\mathbf{X}$ of $A^{\mathbb{Z}}$, we can define the *language* of $\mathbf{X}$ as the set of words that appear in some element of $\mathbf{X}$: $\mathcal{L}(\mathbf{X}) = \left\{w \in A^*, \exists x \in \mathbf{X}, \exists i \in \mathbb{Z}, x_{[i;i+|w|-1]} = w\right\}$. To ease notations we denote, for $x \in A^{\mathbb{Z}}$, we denote $\mathcal{L}(x)$ for $\mathcal{L}(\{x\})$. When $w \in \mathcal{L}(\mathbf{X})$, we say that w is an $\mathbf{X}$-*word*. Denote by $\mathcal{L}_n(\mathbf{X})$ the set of words of length n appearing in $\mathbf{X}$, *i.e.*, $\mathcal{L}_n(\mathbf{X}) = \mathcal{L}(\mathbf{X}) \cap A^n$.

Shift and subshifts Define the *shift* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ as $\sigma(x)_i = x_{i+1}$. σ is bijective, thus induces a $\mathbb{Z}$-action on the fullshift $A^{\mathbb{Z}}$. A subset $\mathbf{X}$ of $A^{\mathbb{Z}}$ is said to be *shift-invariant* if $\sigma(\mathbf{X}) = \mathbf{X}$. A *subshift* of $A^{\mathbb{Z}}$ is a *closed and shift-invariant* subset of $A^{\mathbb{Z}}$.

Transitive and asymptotic configurations For a subshift $\mathbf{X}$, a configuration $x \in \mathbf{X}$ is said to be *left-transitive* in $\mathbf{X}$ if $\mathcal{O}_-(x) = \left\{\sigma^i(x), i \leq 0\right\}$ is dense in $\mathbf{X}$. It is *right-transitive* in $\mathbf{X}$ if $\mathcal{O}_+(x) = \left\{\sigma^i(x), i \geq 0\right\}$ is dense in $\mathbf{X}$. Two configurations $x, y \in A^{\mathbb{Z}}$ are said to be *left-asymptotic* if there exists $n \in \mathbb{Z}$ such that for all $i \leq n$, $x_i = y_i$. They are *right-asymptotic* if there exists $n \in \mathbb{Z}$ such that for all $i \geq n$, $x_i = y_i$.

Forbidden words It is well known that a subshift can also be defined by a set of *forbidden words* $\mathcal{F} \subseteq A^*$: $\mathbf{X}$ is a subshift of $A^{\mathbb{Z}}$ if and only if there exists $\mathcal{F} \subseteq A^*$ such that $\mathbf{X} = \left\{x \in A^{\mathbb{Z}}, \forall w \in \mathcal{F}, w \notin \mathcal{L}(x)\right\}$. The above $\mathcal{F}$ can be always chosen as $A^* \setminus \mathcal{L}(\mathbf{X})$. When such an $\mathcal{F}$ can be chosen *finite* we say that $\mathbf{X}$ is a *subshift of*

finite type, SFT in short. If the length of the longest word of such a finite $\mathcal{F}$ is not greater than 2 then it is said to be a *one-step SFT*.

Factor maps Let $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ be subshifts. A map $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ is *shift-commuting* if $\sigma \circ f = f \circ \sigma$. A continuous, shift-commuting and onto map $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ is called a *factor map*. A bijective factor map is called a *conjugacy*. If there exist factor maps $\pi : \mathbf{X} \to \mathbf{Y}$ and $\varphi : \mathbf{Y} \to \mathbf{X}$ then we say that the subshifts $\mathbf{X}$ and $\mathbf{Y}$ are *weakly conjugate*. A *sofic shift* is the image of an SFT by a factor map. It is clear that a subshift conjugate to an SFT or a sofic shift is itself, respectively, an SFT or a sofic shift. If $\pi : \mathbf{\Sigma} \to \mathbf{X}$ is a factor map from an SFT onto a sofic shift then $(\mathbf{\Sigma}, \pi)$ is called a *cover* of $\mathbf{X}$.

Sliding block codes For D a finite subset of $\mathbb{Z}$, a *block map* on D is a function $g : A^D \to B$ where A and B are finite sets. g defines a *sliding block code* $f : A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ by $f(x)_i = g(x_{|D+i})$. When $D = \{0\}$, f is said to be *one-block*. By the Curtis-Hedlund-Lyndon theorem [**12**], sliding block codes between $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$ are exactly the continuous and shift-commuting maps between those spaces. Among other things, this implies that a bijective sliding block code (*i.e.*, a conjugacy) has a sliding block code inverse.

Magic words Let $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ be a one-block factor map. Let $m = m_1 \dots m_k$ be a $\mathbf{\Gamma}$-word. For $0 < i \leq k$, let $d^*_f(m, i)$ be the number of different symbols we can see at position i in a $f-$pre-image of m, that is: $d^*_f(m, i) = |p_f(m, i)|$ where $p_f(m, i) = \{c_i, f(c) = m\}$. Denote by $c^*(f)$ the minimum of $d^*_f(m, i)$ over all $i \in \mathbb{N}$ and all $\mathbf{\Gamma}$-words m. A word m such that $d^*_f(m, i) = c^*(f)$ is a called a *magic word* for f at coordinate i.

The following property of magic words will help in understanding better the notions we use in this paper:

Proposition 1.1 (Mainly [**14**, Corollary 9.1.10]) *Let $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ be a one-block factor map and m a magic word for f at coordinate i. For any $\mathbf{\Gamma}$-word of the form vmw (that is, an extension of m) and any symbol $c \in p_f(m, i)$ there exists an $f-$pre-image VMW of vmw such that $M_i = c$.*

Proof. Note that any $\mathbf{\Lambda}$-word VMW that is an f-pre-image of vmw is such that $M_i \in p_f(m, i)$, that is $p_f(vmw, i) \subseteq p_f(m, i)$. But since w is magic at coordinate i, we have $|p_f(m, i)| = d^*_f(m, i) \leq d^*_f(vmw, i) = |p_f(vmw, i)|$, hence $p_f(vmw, i) = p_f(m, i)$. $\square$

Entropy For a subshift $\mathbf{X}$, one can define its *entropy*, which roughly speaking represents the exponential growth rate of its language:

$$h(\mathbf{X}) = \lim_{n\to\infty} \frac{\log |\mathcal{L}_n(\mathbf{X})|}{n}$$

$$
\begin{tikzcd}
 & \mathbf{F} \arrow[ddl, "\rho_1"'] \arrow[ddr, "\rho_2"] \arrow[d, phantom, "\cup|"] & \\
 & \mathbf{F}' \arrow[dl, "\rho_{1|\mathbf{F}'}"] \arrow[dr, "\rho_{2|\mathbf{F}'}"'] & \\
\mathbf{\Sigma}_1 \arrow[dr, "\pi_1"'] & & \mathbf{\Sigma}_2 \arrow[dl, "\pi_2"] \\
 & \mathbf{X} &
\end{tikzcd}
$$

Figure 1. A commutative diagram of the fiber product and full fiber product of $(\mathbf{\Sigma}_1, \pi_1)$ and $(\mathbf{\Sigma}_2, \pi_2)$.

For example, if $\mathbf{X}$ and $\mathbf{Y}$ are conjugate subshifts then they have the same entropy. If $\mathbf{X}$ factors onto $\mathbf{Y}$ then $h(\mathbf{Y}) \leq h(\mathbf{X})$, that is, entropy does not increase *via* factor maps.

Irreducibility and mixing A subshift $\mathbf{X}$ is said to be *irreducible* if for any two configurations $x, y \in \mathbf{X}$, there exists $N \in \mathbb{N}$ and $z \in \mathbf{X}$ such that $z_i = x_i$ for $i \leq 0$ and $z_i = y_i$ for $i \geq N$. It is well known that if $\mathbf{X}$ is sofic then there exists such an N that does not depend on the configurations x and y. $\mathbf{X}$ is said to be *mixing* if there exists $N \in \mathbb{N}$ such that for any $k \geq N$ and any two configurations $x, y \in \mathbf{X}$ there exists $z \in \mathbf{X}$ such that $z_i = x_i$ for $i \leq 0$ and $z_i = y_i$ for $i \geq k$. A factor of an irreducible subshift is itself irreducible and a factor of a mixing subshift is also mixing.

Fiber product A classical construction from symbolic dynamics [**14**, Definition 8.3.2] is the *fiber product* of two covers of the same sofic shift: Let $(\mathbf{\Sigma}_1, \pi_1)$ and $(\mathbf{\Sigma}_2, \pi_2)$ be covers of the same sofic shift $\mathbf{X}$. We define the *full fiber product* $\mathbf{F}$ of $(\mathbf{\Sigma}_1, \pi_1)$ and $(\mathbf{\Sigma}_2, \pi_2)$ as: $\mathbf{F} = \{(x_1, x_2), x_1 \in \mathbf{\Sigma}_1, x_2 \in \mathbf{\Sigma}_2, \pi_1(x_1) = \pi_2(x_2)\}$. $\mathbf{F}$ comes with canonical projections: $\rho_1 : \mathbf{F} \to \mathbf{\Sigma}_1, \rho_1(x_1, x_2) = x_1$ and $\rho_2 : \mathbf{F} \to \mathbf{\Sigma}_2, \rho_1(x_1, x_2) = x_2$. Usually, ρ_1 inherits the properties of π_2 and ρ_2 those of π_1 [**14**, Proposition 8.3.3]; we will state precisely what this means when we will need it. Since both $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are SFTs, so is $\mathbf{F}$. If $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ are irreducible, then $\mathbf{F}$ is not necessarily irreducible, however, it contains a unique irreducible component of maximal entropy $\mathbf{F}'$ and the restrictions of ρ_1 and ρ_2 to $\mathbf{F}'$ remain surjective. $\mathbf{F}'$ is called the *fiber product* of $(\mathbf{\Sigma}_1, \pi_1)$ and $(\mathbf{\Sigma}_2, \pi_2)$. The situation after all those definitions is depicted on Figure 1.

Right-closing and resolving A factor map $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ between subshifts is said to be *right-closing* if it never collapses two left-asymptotic points. That is:

$$\forall x, y \in \mathbf{\Lambda}, \forall i \leq 0, x_i = y_i, f(x) = f(y) \Rightarrow x = y$$

f is *right-closing almost-everywhere* if we only require the above to hold for left-transitive x and y. If f is a one-block map, it is said to be *right-resolving* if whenever ab and ac are two-letters words in $\mathbf{\Lambda}$ then $f(b) = f(c)$ implies $b = c$.

Minimal right-resolving cover Among the covers of an irreducible sofic shift $\mathbf{X}$, there is one of particular interest: the *minimal right-resolving cover*, or *Fischer cover* [**11**] $(\mathbf{\Sigma}_R, \pi_R)$. $\pi_R : \mathbf{\Sigma}_R \to \mathbf{X}$ is a one-block right-resolving factor map and if $f : \mathbf{\Sigma} \to \mathbf{X}$ is a right-closing factor map then there exists $\varphi : \mathbf{\Sigma} \to \mathbf{\Sigma}_R$ such that $f = \pi_R \circ \varphi$ [**8**, Proposition 4].

Periodic points A configuration x is said to be *periodic* if there exists an integer $i > 1$ such that $\sigma^i(x) = x$. The *period* of x is the smallest such i. We denote by $\mathrm{Per}(\mathbf{X})$ the *set of periodic points* of the subshift $\mathbf{X}$. If a subshift $\mathbf{X}$ factors onto a subshift $\mathbf{Y}$ then for every periodic point x of $\mathbf{X}$ there exists a periodic point y of $\mathbf{Y}$ whose period divides the period of x (take y to be the image of x by the factor map). We denote this relation $\mathrm{Per}(\mathbf{X}) \to \mathrm{Per}(\mathbf{Y})$. It turns out that this trivial necessary condition on periodic points is also sufficient for the existence of a factor map between two irreducible SFTs of unequal entropy [**5**]. A periodic point x is represented as a finite word w, whose length is the period of x, repeated infinitely: $x = {}^\infty w^\infty$. Following [**5**], for an irreducible sofic shift $\mathbf{X}$ with minimal right-resolving cover $(\mathbf{\Sigma}_R, \pi_R)$, we say that such an x is a *receptive periodic point* if there exist magic words for π_R: m_1 and m_2 such that for every $n \geq 1$, $m_1 w^n m_2$ is an $\mathbf{X}$-word. If $\mathbf{X}$ is SFT then any periodic point is receptive because π_R is a conjugacy. Following [**15**] we say that a configuration x is a *receptive fixed point* if it is a receptive periodic point of period 1. As remarked at the end of section 2 in [**15**], a factor map between irreducible sofic shifts maps receptive fixed points to receptive fixed points.

Right-resolving almost everywhere If $\mathbf{\Lambda}$ is an irreducible sofic shift with minimal right-resolving cover $(\mathbf{\Sigma}_R, \pi_R)$ and f is one-block, we say that f is *right-resolving almost-everywhere* if f is right-closing almost-everywhere and $f \circ \pi_R : \mathbf{\Sigma}_R \to \mathbf{\Gamma}$ is right-resolving.

Right-continuing and right-e-resolving A factor map $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ between sofic shifts is said to be *right-continuing* [**10**] if for any x in $\mathbf{\Lambda}$ and y in $\mathbf{\Gamma}$ such that $f(x)$ and y are left-asymptotic, there exists x' left-asymptotic to x in $\mathbf{\Lambda}$ such that $f(x) = y$. If there exists an integer n such that for any $x \in \mathbf{\Lambda}$ and $y \in \mathbf{\Gamma}$ such that $f(x)_{(-\infty;n]} = y_{(-\infty;n]}$ then there exists $x' \in \mathbf{\Lambda}$ such that $x'_{(-\infty;0]} = x_{(-\infty;0]}$ and $f(x') = y$ then f is said to be *right-continuing with retract n*. If f is right-continuing with retract 0 then it is said to be *right-e-resolving*. As before, we define

right-continuing almost-everywhere and *right-e-resolving almost-everywhere* when we only require the above to hold for a left-transitive y but impose the existence of a retract. Indeed, Proposition 1.1 implies that *any* factor map with an SFT domain is right (and left) continuing almost-everywhere, but without retract. We will usually not append "with a retract" when talking about right-continuing almost-everywhere factor maps and consider the existence of the retract to be part of the definition of right-continuing almost-everywhere. By [**10**, Proposition 5.1, (iii)⇒(i)], if f is right-continuing (resp. right-continuing *a.e.*) with a retract then there exists a conjugacy Θ and f' such that $f = f' \circ \Theta$ and f' is right-e-resolving (resp. right-e-resolving almost-everywhere). Also remark that in the definition of right-closing *a.e.* we imposed x to be left-transitive while in the definition of right-continuing *a.e.* we impose y to be left-transitive: it is simply a matter of historical definitions, right-closing *a.e.* has, to our knowledge, always been defined as such while we could impose y to be left-transitive in the definition of right-closing *a.e.* since for a finite-to-one factor map f, $f(x)$ is left-transitive if and only if x is by a slight modification of [**14**, Lemma 9.1.13].

Links between right-closing and right-continuing with a retract Right-continuing shall be seen as the dual of right-closing and right-e-resolving the dual of right-resolving. One may remark that the above definition of right-e-resolving is more intricate than the original one for SFTs in [**10**] and than its right-resolving dual; they are equivalent when $\mathbf{\Lambda}$ is SFT but differ when it is merely sofic: With the original definition we may have right-e-resolving factor maps over sofic shifts which are not right-continuing [**22**]. The above definition avoids this problem and is equivalent to the original one for SFTs by [**10**, Proposition 5.1].

While the right-continuing image of an SFT is an SFT [**22**] (or [**6**, Proposition 2.1] for the finite-to-one case, or even [**14**, Proposition 8.2.2]), a right-closing factor map from an SFT is right-continuing almost-everywhere [**6**, Lemma 2.5]. A right-closing almost-everywhere factor map with SFT domain is right-closing (everywhere) [**9**, Proposition 4.10]. Therefore for a finite-to-one f, we may ask whether right-closing almost-everywhere is equivalent to right-continuing almost-everywhere.

Proposition 1.2 (Mainly [**6**, Lemma 2.5]) *A factor map $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ between irreducible sofic shifts which is right-closing almost-everywhere is also right-continuing almost-everywhere (with a retract). If f is right-resolving almost-everywhere then it is right-e-resolving almost-everywhere.*

Proof. Let $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ be right-closing almost-everywhere. Let $(\mathbf{\Sigma}_R, \pi_R)$ be the minimal right-resolving cover of $\mathbf{\Lambda}$. $f \circ \pi_R : \mathbf{\Sigma}_R \to \mathbf{\Gamma}$ is right-closing almost-everywhere [**9**, Proposition 4.11] and thus right-closing [**9**, Proposition 4.10]. By [**6**, Lemma 2.5], $f \circ \pi_R$ is right-continuing almost-everywhere with a retract. Let x be left-transitive in $\mathbf{\Lambda}$ and $\tilde{x}$ its (left-transitive) pre-image in $\mathbf{\Sigma}_R$. Let y be left-asymptotic to $f(x)$ in $\mathbf{\Gamma}$. Since $f \circ \pi_R$ is right-continuing almost-everywhere, find $\tilde{x}'$ in $\mathbf{\Sigma}_R$, left-asymptotic to $\tilde{x}$ such that $f \circ \pi_R(\tilde{x}') = y$. $x' = \pi_R(\tilde{x}')$ is the x' we were

looking for. The right-resolving case follows similarly to [**6**, Lemma 2.5]: x'_{i+1} and x_{i+1} are uniquely determined by y_{i+1} and, respectively, x'_i and x_i; since $y_i = f(x_i)$ for $i \leq 0$ and $x'_i = x_i$ for $i \leq -n$, then $x'_i = x_i$ for $i \leq 0$. □

The converse of Proposition 1.2 holds when f is finite-to-one:

PROPOSITION 1.3. *If a finite-to-one factor map $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ between irreducible sofic shifts is right-continuing* a.e. *(with a retract) then it is right-closing almost-everywhere.*

Proof. Up to a conjugacy we can assume that f is right-e-resolving almost-everywhere. Let $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ be right-e-resolving almost-everywhere and suppose it is not right-closing almost-everywhere. Since, from Proposition 1.2, π_R is right-e-resolving almost-everywhere, $f \circ \pi_R : \mathbf{\Sigma}_R \to \mathbf{\Gamma}$ is also right-e-resolving almost-everywhere. By [**9**, Proposition 4.11], $f \circ \pi_R$ is right-closing almost-everywhere if and only if f is. Therefore, by considering $f \circ \pi_R$ we can assume that $\mathbf{\Lambda}$ is a one-step SFT. Since f is finite-to-one, by [**14**, Proposition 9.1.7], we may assume that f has a magic symbol b.

Let x and y be two left-transitive left-asymptotic configurations of $\mathbf{\Lambda}$ such that $f(x) = f(y) = z$. Without loss of generality, suppose $x_i = y_i$ for all $i < 0$ and $x_0 \neq y_0$. By irreducibility of $\mathbf{\Gamma}$, let z' be a right-transitive configuration of $\mathbf{\Gamma}$ such that for all $i \leq 0$, $z_i = z'_i$. Since f is right-e-resolving almost-everywhere, let x' and y' be configurations of $\mathbf{\Lambda}$ such that $f(x') = f(y') = z'$ and for all $i \leq 0$, $x_i = x'_i$ and $y_i = y'_i$. Since z' is bi-transitive, let $j < 0$ and $k > 0$ be such that $z'_j = z'_k = b$. $x'_{[j;k]}$ and $y'_{[j;k]}$ are two pre-images of a word starting and ending by the magic symbol b, therefore they are mutually separated by [**14**, Proposition 9.1.9]: they are either equal or differ in every coordinate. However, $x'_{-1} = x_{-1} = y_{-1} = y'_{-1}$ and $x'_0 = x_0 \neq y_0 = y'_0$ and since -1 and 0 are in the interval $[j;k]$, $x'_{[j;k]}$ and $y'_{[j;k]}$ cannot be mutually separated, a contradiction. □

Note that we cannot remove the hypothesis on the retract in Proposition 1.3: Otherwise since Proposition 1.1 implies that *any* factor map is right-continuing almost-everywhere without retract, any finite-to-one factor map from an SFT would be right-closing almost-everywhere and thus right-closing by [**9**, Proposition 4.10], however there exist finite-to-one factor maps between SFTs that are not right-closing.

The following proposition shall be seen as the dual of [**9**, Proposition 4.12] which states that a right-closing *a.e.* factor map from an SFT onto a sofic shift is right-closing everywhere:

PROPOSITION 1.4. *A right-e-resolving almost-everywhere factor map $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$, where $\mathbf{\Lambda}$ is an irreducible sofic shift and $\mathbf{\Gamma}$ is an irreducible SFT, is right-e-resolving (everywhere).*

Proof. Let $x \in \mathbf{\Lambda}$ and $y \in \mathbf{\Gamma}$ be such that $f(x)_i = y_i$ for all $i \leq 0$. For an integer n, find by irreducibility of $\mathbf{\Lambda}$ a left-transitive configuration $x^n \in \mathbf{\Lambda}$ such that $x^n_i = x_i$ for all $i \geq -n$. Define y^n such that $y^n_i = y_i$ for $i \geq 0$ and $y^n_i = f(x^n)_i$ for $i < 0$.

For n sufficiently big, y^n belongs to $\mathbf{\Gamma}$ since it is SFT. y^n is left-transitive because x^n is and f is onto.

Since f is right-e-resolving almost-everywhere, find z^n such that $f(z^n) = y^n$ and $z^n_i = x^n_i$ for $i \le 0$. By compactness of $\mathbf{\Lambda}$, we may assume w.l.o.g. that z^n converges to $z \in \mathbf{\Lambda}$. y^n clearly converges to y, thus, by continuity of f, $f(z) = y$. Moreover, for all $i \le 0$, $z_i = x_i$, thus f is right-e-resolving everywhere. □

Again, in Proposition 1.4, right-e-resolving almost-everywhere can be replaced by right-continuing almost-everywhere with a retract and we get a right-continuing with a retract factor in the conclusion. Without the retract hypothesis, it may be possible that the z^n we find agrees with x^n only at positions $i < -n$ so that its limit may not be left-asymptotic to x at all.

Proposition 1.5. *Let $\Phi : \mathbf{X} \to \mathbf{Y}$ and $\Psi : \mathbf{Y} \to \mathbf{Z}$ be factor maps between irreducible sofic shifts. If $\Psi \circ \Phi : \mathbf{X} \to \mathbf{Z}$ is right-continuing almost-everywhere with a retract then so is Ψ.*

Proof. Let N be the retract of $\Psi \circ \Phi$. Assume without loss of generality that Ψ and Φ are both one-block. Let $y \in \mathbf{Y}$ be left-transitive and $z \in \mathbf{Z}$ be such that $\Psi(y)_i = z_i$ for $i \le N$. Let $x \in \mathbf{X}$ be a left-transitive pre-image of y. By our hypothesis, there exists $x' \in \mathbf{X}$ such that $x'_i = x_i$ for $i \le 0$ and $\Psi \circ \Phi(x') = y$. Let $y' = \Phi(x')$. Since Φ is one-block, $y'_i = y_i$ for $i \le 0$. And $\Psi(y') = z$, thus Ψ is right-continuing almost-everywhere with retract N. □

Followers Let $\mathbf{\Sigma}$ be a one-step SFT and $f : \mathbf{\Sigma} \to \mathbf{\Gamma}$ be a one-block factor map onto a sofic shift. For a letter a of the alphabet of $\mathbf{\Sigma}$, denote $\mathcal{F}^{\mathbf{\Sigma}}_f(a)$ the *$f-$follower* of a: $\mathcal{F}^{\mathbf{\Sigma}}_f(a) = \{f(aw), aw \in \mathcal{L}(\mathbf{\Sigma})\}$. $(\mathbf{\Sigma}, f)$ is said to be *follower separated* [**14**, Definition 3.3.7] if for any letters a and b of the alphabet of $\mathbf{\Sigma}$, if $\mathcal{F}^{\mathbf{\Sigma}}_f(a) = \mathcal{F}^{\mathbf{\Sigma}}_f(b)$ then $a = b$.

Remark that the minimal right-resolving cover of a sofic shift is always follower separated [**14**, Proposition 3.3.9], it is actually the only (up to conjugacy) cover that is both follower separated and right-resolving. It is also well known that any factor map from an SFT can be decomposed through a follower separated factor map with SFT domain:

Lemma 1.6 ([**14**, Section 3.3] or the remarks before [**19**, Proposition 1.2]) *Let $\mathbf{\Sigma}$ be a one-step SFT and $f : \mathbf{\Sigma} \to \mathbf{\Gamma}$ a one-block factor map onto a sofic shift. There exists a one-step SFT $\tilde{\mathbf{\Sigma}}$ and one-block factor maps $\varphi : \mathbf{\Sigma} \to \tilde{\mathbf{\Sigma}}$ and $\pi : \tilde{\mathbf{\Sigma}} \to \mathbf{\Gamma}$ such that $f = \pi \circ \varphi$ and $(\tilde{\mathbf{\Sigma}}, \pi)$ is follower-separated.*

In [**19**, Proposition 1.2], it is proved, in addition, that when f is finite-to-one then φ can be chosen right-resolving.

Cellular automata and their limit set A *cellular automaton* is a sliding block code $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, *i.e.*, an endomorphism of a fullshift. The *limit set* of a cellular

$$
\begin{tikzcd}
\mathbf{\Sigma} \arrow[dr, "f"] & \\
\mathbf{\Lambda} \arrow[r, "f"'] & \mathbf{\Gamma}
\end{tikzcd}
$$

Figure 2. A steady factor map.

automaton f is denoted by Ω_f and is defined by:

$$\Omega_f = \bigcap_{n\in\mathbb{N}} f^n(A^{\mathbb{Z}})$$

One can prove by a simple compactness argument that Ω_f is precisely the set of configurations that have f^n-pre-images for any integer n. A cellular automaton is said to be *stable* [**15**] if there exists an integer N such that $\Omega_f = f^N(A^{\mathbb{Z}})$; in other words the cellular automaton reaches its limit set in finitely many steps. Ω_f is always closed and shift-invariant, hence a subshift. Since $A^{\mathbb{Z}}$ is a mixing sofic shift with a receptive fixed point, so is $\Omega_f = f^N(A^{\mathbb{Z}})$ when f is a stable cellular automaton. These are the necessary conditions for being a stable limit set of cellular automaton that A. Maass called property (H) in [**15**]. In [**15**] he also proved that these conditions happen to be sufficient for almost of finite type shifts (see Section 4 for the definition and more details on these sofic shifts). However, it is an important open problem to get a characterization of such subshifts that can occur as limit sets of cellular automata in the general case, even for the, *a priori*, simpler case where we assume the cellular automata to be stable [**7**, Section 16].

Steady maps Let $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ be irreducible sofic shifts. A factor map $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ is said to be *steady* [**4**], or $\mathbf{\Sigma}$-steady, if there exists an SFT $\mathbf{\Sigma}$ containing $\mathbf{\Lambda}$ such that f is well defined on $\mathbf{\Sigma}$ and $f(\mathbf{\Sigma}) = f(\mathbf{\Lambda}) = \mathbf{\Gamma}$. The diagram of a steady map is represented on Figure 2.

Steady maps provide a good formalism for stable limit set of cellular automata: if $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a stable cellular automaton, then there exists N such that $f^N(A^{\mathbb{Z}}) = \Omega_f$. By definition of Ω_f, we have $f^N(\Omega_f) = \Omega_f$, thus, $f^N : \Omega_f \to \Omega_f$ is a steady epimorphism of Ω_f. This means that stable cellular automata are a special kind of steady maps between irreducible sofic shifts. In the rest of the paper we will focus on steady maps and state the results we obtain for them as theorems while their implications on stable limit set of cellular automata will be corollaries, even if characterizing limit sets of stable cellular automata is what motivated our study.

If $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ is a $\mathbf{\Sigma}$-steady factor map, then we say that f is a *right-closing almost-everywhere steady map* if $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ is, in addition, right-closing almost-everywhere. f is a *right-continuing almost-everywhere steady map* if $f : \mathbf{\Sigma} \to \mathbf{\Gamma}$

is right-continuing almost-everywhere (with a retract). Note that the domain on which we consider f differs between the two definitions. The former class of steady maps is studied in Section 2 and the latter in Section 3.

For a stable cellular automaton $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, let N be an integer such that $f^N(A^{\mathbb{Z}}) = \Omega_f$. We say that f is a *right-closing almost-everywhere cellular automaton* when f^N is a right-closing almost-everywhere steady map, f is a *right-continuing almost-everywhere stable cellular automaton* when f^N is a right-continuing almost-everywhere steady map.

2 Right-closing almost everywhere steady maps

In this section we study right-closing almost-everywhere steady maps. We prove that such maps can always be decomposed through the minimal right-resolving cover of its range (Lemma 2.1) and then characterize sofic shifts between which there can exist such a factor (Theorem 2.2) so that we get a characterization of sofic shifts that are stable limit set of cellular automata with a right-closing *a.e.* dynamics on its limit set (Corollary 2.3).

Lemma 2.1. *If $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ is a right-closing almost-everywhere $\mathbf{\Sigma}$-steady factor map then f can be decomposed through the minimal right-resolving cover of $\mathbf{\Gamma}$ by a right-continuing factor map with a retract,* i.e., *there exists $\varphi : \mathbf{\Lambda} \to \mathbf{\Sigma}_R$ such that $f = \pi_R \circ \varphi$ and φ is right-continuing with a retract.*

Proof. Without loss of generality (up to a conjugacy), we can assume that f is one-block and that $\mathbf{\Sigma}$ is the one-step SFT approximation of $\mathbf{\Lambda}$ such that $f(\mathbf{\Sigma}) = f(\mathbf{\Lambda}) = \mathbf{\Gamma}$.

Let φ and π be the factor maps given by applying Lemma 1.6 to $(\mathbf{\Sigma}, f)$. Let $\tilde{\mathbf{\Lambda}} = \varphi(\mathbf{\Lambda})$ and $\tilde{\mathbf{\Sigma}} = \varphi(\mathbf{\Sigma})$. This is summarized in the following diagram:

$$\begin{tikzcd}
\mathbf{\Sigma} \arrow[r, "\varphi"] & \tilde{\mathbf{\Sigma}} \arrow[dr, "\pi"] & \\
\cup\!\mid & \cup\!\mid & \\
\mathbf{\Lambda} \arrow[r, "\varphi"'] & \tilde{\mathbf{\Lambda}} \arrow[r, "\pi"'] & \mathbf{\Gamma}
\end{tikzcd}$$

By [**9**, Proposition 4.11], both $\varphi : \mathbf{\Lambda} \to \tilde{\mathbf{\Lambda}}$ and $\pi : \tilde{\mathbf{\Lambda}} \to \mathbf{\Gamma}$ are right-closing almost-everywhere. Without loss of generality, we can assume that they are right-resolving almost-everywhere. By Proposition 1.2, $\pi : \tilde{\mathbf{\Lambda}} \to \mathbf{\Gamma}$ is right-e-resolving almost-everywhere.

Claim 1. $\pi : \tilde{\mathbf{\Lambda}} \to \mathbf{\Gamma}$ *is right-resolving (everywhere).*

Proof of Claim 1. Suppose that we have ab_1 and ab_2 allowed in $\tilde{\mathbf{\Lambda}}$ such that $\pi(b_1) = \pi(b_2) = b$. Since $(\tilde{\mathbf{\Sigma}}, \pi)$ is follower separated, $\mathcal{F}_{\pi}^{\tilde{\mathbf{\Sigma}}}(b_1) \neq \mathcal{F}_{\pi}^{\tilde{\mathbf{\Sigma}}}(b_2)$. Without loss of generality, let w be a word such that $b_1 w$ is allowed in $\tilde{\mathbf{\Sigma}}$ and

$\pi(w) \in \mathcal{F}_\pi^{\tilde{\mathbf{\Sigma}}}(b_1) \setminus \mathcal{F}_\pi^{\tilde{\mathbf{\Sigma}}}(b_2)$. Let z be a right-transitive word of $\tilde{\mathbf{\Sigma}}$ starting by w and xab_2y be a left-transitive configuration of $\tilde{\mathbf{\Lambda}}$; xab_1z is a valid configuration of $\tilde{\mathbf{\Sigma}}$ since it is a one-step SFT, thus, $\pi(xab_1z) = \pi(xa)\pi(z)$ is a configuration of $\mathbf{\Gamma}$. Since $\pi : \tilde{\mathbf{\Lambda}} \to \mathbf{\Gamma}$ is right-e-resolving almost-everywhere, there exists z' such that xab_2z' is a configuration of $\tilde{\mathbf{\Lambda}}$ and $\pi(xab_2z') = \pi(xa)\pi(z) = \pi(xa)\pi(w)\pi(z)_{[|w|;\infty)}$. However, we assumed that $\pi(w) \notin \mathcal{F}_\pi^{\tilde{\mathbf{\Sigma}}}(b_2)$, a contradiction. We therefore conclude that such ab_1 and ab_2 cannot exist and thus that π is right-resolving everywhere. □

Since $\pi : \tilde{\mathbf{\Lambda}} \to \mathbf{\Gamma}$ is right-closing and $\pi(\tilde{\mathbf{\Sigma}}) = \mathbf{\Gamma}$, by [**15**, proof of Proposition 4.2], $\tilde{\mathbf{\Lambda}}$ is SFT. By [**8**, Proposition 4], π being a right-closing cover of $\mathbf{\Gamma}$ can be decomposed through $\mathbf{\Sigma}_R$, which proves the existence of the decomposition: $f = \pi_R \circ \varphi$.

Now, since f is right-closing almost-everywhere, so is φ by [**9**, Proposition 4.11]. By taking a conjugacy, we may assume that φ is right-resolving almost-everywhere, but φ maps $\mathbf{\Lambda}$ onto $\mathbf{\Sigma}_R$ which is SFT, thus by Proposition 1.4, φ is right-e-resolving everywhere. By unwinding the conjugacy we took at the beginning, we get that the original φ is right-continuing with a retract. □

Theorem 2.2. *Let* $\mathbf{\Lambda}$ *and* $\mathbf{\Gamma}$ *be irreducible sofic shifts of equal entropy. There exists a right-closing* a.e. *steady factor map from* $\mathbf{\Lambda}$ *onto* $\mathbf{\Gamma}$ *if and only if there exists a right-continuing (with a retract) factor map from* $\mathbf{\Lambda}$ *onto the minimal right-resolving cover of* $\mathbf{\Gamma}$.

Proof. $\Rightarrow$: Lemma 2.1.

$\Leftarrow$: Let $\varphi : \mathbf{\Lambda} \to \mathbf{\Sigma}_R$ be a right-continuing factor map from $\mathbf{\Lambda}$ onto the minimal right-resolving cover of $\mathbf{\Gamma}$. $\mathbf{\Lambda}$ and $\mathbf{\Sigma}_R$ having the same entropy, φ is finite-to-one and thus right-closing *a.e.* by Proposition 1.3. $\pi_R \circ \varphi$ is thus a right-closing almost-everywhere factor map from $\mathbf{\Lambda}$ onto $\mathbf{\Gamma}$ by [**9**, Proposition 4.11].

It remains to prove that $\pi_R \circ \varphi$ is steady. Let $\mathbf{\Sigma}$ be an irreducible SFT containing $\mathbf{\Lambda}$ such that $\mathrm{Per}(\mathbf{\Sigma}) \to \mathrm{Per}(\mathbf{\Sigma}_R)$. Note that such an SFT $\mathbf{\Sigma}$ always exists by *e.g.*, [**2**, Lemma 4.1]. Now we can apply Boyle's extension lemma [**5**, Lemma 2.4] to extend φ to $\tilde{\varphi} : \mathbf{\Sigma} \to \mathbf{\Sigma}_R$. $f = \pi_R \circ \tilde{\varphi}$ is therefore the desired right-closing *a.e.* steady factor map. □

Corollary 2.3. *A subshift is the stable limit set of a cellular automaton which has a right-closing* a.e. *dynamics on its limit set if and only if it is a factor of a fullshift*† *and factors by a right-continuing factor map onto its minimal right-resolving cover.*

3 Right-continuing almost-everywhere steady maps

In this section we study right-continuing almost-everywhere steady maps. It is organized the same way as Section 2: We prove that such maps can always be

† Being a factor of a fullshift is equivalent to the property (H) of A. Maass in [**15**] by Boyle's lower entropy factor theorem for sofic shifts [**5**, Theorem 3.3].

decomposed through the minimal right-resolving cover of its range (Lemma 3.1) and then characterize sofic shifts between which there can exist such a factor (Theorem 3.2) so that we get a characterization of sofic shifts that are stable limit set of cellular automata that attain their limit set with a right-continuing almost-everywhere factor map (Corollary 3.3).

Lemma 3.1. *A factor map $f : \mathbf{\Sigma} \to \mathbf{\Gamma}$ that is right-continuing almost-everywhere with retract N from an irreducible SFT $\mathbf{\Sigma}$ onto a sofic shift $\mathbf{\Gamma}$ can be decomposed through the minimal right-resolving cover $(\mathbf{\Sigma}_R, \pi_R)$ of $\mathbf{\Gamma}$.*

Proof. Without loss of generality we can assume that $\mathbf{\Sigma}$ is a one-step SFT and f is one-block.

Let $\mathbf{F}$ be the irreducible component of maximal entropy of the fiber-product of $(\mathbf{\Sigma}, f)$ and $(\mathbf{\Sigma}_R, \pi_R)$. Let $\rho_1 : \mathbf{F} \to \mathbf{\Sigma}$ and $\rho_2 : \mathbf{F} \to \mathbf{\Sigma}_R$ be the canonical projections. By [**18**, Proposition 5.1] f can be decomposed through $(\mathbf{\Sigma}_R, \pi_R)$ if and only if ρ_1 is a conjugacy.

Suppose ρ_1 is not a conjugacy: Let (x, y^1) and (x, y^2) be two configurations of $\mathbf{F}$ that have the same ρ_1-image. Without loss of generality, suppose $y^1_N \neq y^2_N$. Since y^1_N and y^2_N are symbols of $\mathbf{\Sigma}_R$ and π_R is follower-separated, we can assume that there exists w such that $y^1_N w$ is allowed in $\mathbf{\Sigma}_R$ and no $y^2_N w'$, where $\pi_R(w') = \pi_R(w)$, is allowed.

By irreducibility of $\mathbf{F}$, find a left-transitive configuration $(x', y') \in \mathbf{F}$ such that $(x'_i, y'_i) = (x_i, y^1_i)$ for $i \geq 0$. Let $y'' \in \mathbf{\Sigma}_R$ be such that $y''_i = y'_i$ for $i \leq N$ and $y''_{[N+1;N+|w|]} = w$. Since f is right-continuing almost-everywhere with retract N, there exists x'' such that $f(x'') = \pi_R(y'')$ and $x''_i = x'_i$ for $i \leq 0$. Hence, $x''_0 = x_0$.

Since $x''_0 = x_0$ and $\mathbf{\Sigma}$ is one-step, there exists $z \in \mathbf{\Sigma}$ such that $z_i = x_i$ for $i \leq 0$ and $z_i = x''_i$ for $i > 0$. Since π_R is right-resolving, by classical fiber-product arguments [**14**, Proposition 8.3.3], ρ_1 is also right-resolving. By [**6**, Proposition 2.1(5)], since both $\mathbf{F}$ and $\mathbf{\Sigma}$ are irreducible, ρ_1 is right-e-resolving, and thus there exists y^3 such that $(z, y^3) \in \mathbf{F}$ and $y^3_i = y^2_i$ for $i \leq 0$. Now, $\pi_R(y^3) = f(z)$ and $f(z)_i = \pi_R(y^2)_i$ for $i \leq N$, hence since ρ_1 is right-resolving, $y^3_i = y^2_i$ for $i \leq N$. Let $w' = y^3_{[N+1;N+|w|]}$. Since $f(z) = \pi_R(y^3)$ and $f(z)_{[N+1;N+|w|]} = \pi_R(w)$ we have w' such that $y^2_N w'$ is allowed in $\mathbf{\Sigma}_R$ and $\pi_R(w') = \pi_R(w)$, a contradiction. □

Theorem 3.2. *Let $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ be irreducible sofic shifts. There exists a right-continuing almost-everywhere steady factor map $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ if and only if $\mathbf{\Lambda}$ factors onto $\mathbf{\Sigma}_R$, the minimal right-resolving cover of $\mathbf{\Gamma}$.*

Proof. $\Rightarrow$: If $f : \mathbf{\Sigma} \to \mathbf{\Gamma}$ is right-continuing almost-everywhere with a retract then by Lemma 3.1, there exists $\varphi : \mathbf{\Sigma} \to \mathbf{\Sigma}_R$, where $(\mathbf{\Sigma}_R, \pi_R)$ is the minimal right-resolving cover of $\mathbf{\Gamma}$ such that $f = \pi_R \circ \varphi$. Then, $\varphi(\mathbf{\Lambda}) \subseteq \mathbf{\Sigma}_R$. But then $f(\mathbf{\Lambda}) = \pi_R(\varphi(\mathbf{\Lambda})) = \mathbf{\Gamma}$, and π_R is finite-to-one so that $\varphi(\mathbf{\Lambda})$ and $\mathbf{\Gamma}$ have the same entropy. Hence, $\varphi(\mathbf{\Lambda})$ and $\mathbf{\Sigma}_R$ also have the same entropy, and since $\varphi(\mathbf{\Lambda})$ and $\mathbf{\Sigma}_R$ are both irreducible sofic shifts, they are actually equal: $\varphi : \mathbf{\Lambda} \to \mathbf{\Sigma}_R$ is onto.

$\Leftarrow$: Let $\varphi : \mathbf{\Gamma} \to \mathbf{\Sigma}_R$ be the factor map from $\mathbf{\Gamma}$ onto $\mathbf{\Sigma}_R$ and $\pi_R : \mathbf{\Sigma}_R \to \mathbf{\Gamma}$ be the minimal right-resolving cover of $\mathbf{\Gamma}$. Let $\mathbf{\Sigma}$ be an irreducible SFT containing $\mathbf{\Lambda}$ such that $\mathrm{Per}(\mathbf{\Sigma}) \to \mathrm{Per}(\mathbf{\Sigma}_R)$. Note that such an SFT $\mathbf{\Sigma}$ always exists by [**2**, Lemma 4.1].

By [**10**, Theorem 5.3], φ can be extended to a right-continuing factor map $\tilde{\varphi} : \mathbf{\Sigma} \to \mathbf{\Sigma}_R$. $\tilde{\varphi}$ has a retract and $\pi_R : \mathbf{\Sigma}_R \to \mathbf{\Gamma}$ is right-e-resolving almost-everywhere by Proposition 1.2 or [**6**, Lemma 2.5]. Let N be the retract of $\tilde{\varphi} : \mathbf{\Sigma} \to \mathbf{\Sigma}_R$.

Let $f = \pi_R \circ \tilde{\varphi}$. It is clear that $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ is a $\mathbf{\Sigma}$-steady factor map. We claim that $f : \mathbf{\Sigma} \to \mathbf{\Gamma}$ is right-continuing almost-everywhere with retract N: Let $x \in \mathbf{\Sigma}$ and $y \in \mathbf{\Gamma}$ be a left-transitive configuration such that $f(x)_i = y_i$ for $i \leq N$. Let $x' = \tilde{\varphi}(x) \in \mathbf{\Sigma}_R$ and $y' \in \mathbf{\Sigma}_R$ a π_R-pre-image of y: We have $\pi_R(x') = f(x)$ and $\pi_R(y') = y$. Since $f(x)$ and y are left-transitive and π_R is 1-1 *a.e.* right-resolving, it is clear that $x'_i = y'_i$ for $i \leq N$. Now, since $\tilde{\varphi}$ is right-continuing with retract N, there exists $z \in \mathbf{\Sigma}$ such that $\tilde{\varphi}(z) = y'$ and $z_i = x_i$ for $i \leq 0$. Then $f(z) = \pi_R(\tilde{\varphi}(z)) = \pi_R(y') = y$ and f is indeed right-continuing almost-everywhere with retract N. □

COROLLARY 3.3. *A subshift is the stable limit set of a right-continuing almost-everywhere cellular automaton if and only if it is a factor of a fullshift and is weakly conjugate to its minimal right-resolving cover.*

4 AFT shifts

In this section, we continue with the same methods we used in the previous two sections to obtain a characterization of AFT shifts by means of the type of steady maps that have them as range (Theorem 4.2). As is usually the case, the situation is much simpler in the AFT case and the conclusions can be strengthened.

An irreducible sofic shift $\mathbf{\Gamma}$ is said to be *AFT*, for *Almost of Finite Type*, if its minimal right-resolving cover $(\mathbf{\Sigma}_R, \pi_R)$ is also left-closing [**16**].

LEMMA 4.1. *Let $f : \mathbf{\Sigma} \to \mathbf{\Gamma}$ be a factor map from an irreducible SFT $\mathbf{\Sigma}$ onto a sofic shift $\mathbf{\Gamma}$. If f is right and left-continuing almost-everywhere with a bi-retract then $\mathbf{\Gamma}$ is AFT.*

Proof. By Lemma 3.1, f can be decomposed through $(\mathbf{\Sigma}_R, \pi_R)$. Then, by Proposition 1.5 (and its analogous replacing right-continuing by left-continuing), π_R is right and left-continuing almost-everywhere with a bi-retract. Then, by Proposition 1.3, π_R being finite-to-one is right and left-closing almost everywhere. Finally, by [**9**, Proposition 4.10], since $\mathbf{\Sigma}_R$ is SFT, π_R is both right and left-closing everywhere and thus $\mathbf{\Gamma}$ is AFT. □

THEOREM 4.2. *Let $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ be irreducible sofic shifts. There exists a left and right-continuing almost-everywhere steady factor map $f : \mathbf{\Lambda} \to \mathbf{\Gamma}$ if and only if $\mathbf{\Lambda}$ factors onto the minimal right-resolving cover of $\mathbf{\Gamma}$ and $\mathbf{\Gamma}$ is AFT.*

Proof. $\Rightarrow$: Lemma 4.1.

$\Leftarrow$: Let $\varphi : \mathbf{\Lambda} \to \mathbf{\Sigma}_R$ be the factor map in the hypothesis from $\mathbf{\Lambda}$ onto the minimal right-resolving cover of $\mathbf{\Gamma}$. Let $\mathbf{\Sigma}$ be an irreducible SFT containing $\mathbf{\Lambda}$ such that $\mathrm{Per}(\mathbf{\Sigma}) \to \mathrm{Per}(\mathbf{\Sigma}_R)$. As before, such an SFT $\mathbf{\Sigma}$ always exists by [**2**, Lemma 4.1]. By [**13**, Theorem 4.5], φ can be extended to a bi-continuing factor map $\tilde{\varphi} : \mathbf{\Sigma} \to \mathbf{\Sigma}_R$. Let $f' = \pi_R \circ \tilde{\varphi} : \mathbf{\Sigma} \to \mathbf{\Gamma}$. Since $\mathbf{\Gamma}$ is AFT, π_R is right and left-continuing almost-everywhere with a bi-retract, therefore so is f' as the composition of two such maps. □

Corollary 4.3. *A subshift is the stable limit set of a left and right-continuing almost-everywhere cellular automaton if and only if it is a factor of a fullshift and is AFT.*

Proof. The $\Rightarrow$ direction is clear from Theorem 4.2. The $\Leftarrow$ direction requires a bit more work: By [**15**, Lemma 4.9], the minimal right-resolving cover of an AFT shift that is a factor of a fullshift has a fixed point for σ. By [**2**, Corollary 1.3], there exists a factor map from the AFT onto its minimal right-resolving cover since the condition on periodic points is fulfilled by the existence of a fixed point in the cover. Then we can apply Theorem 4.2. □

5 Conclusions and questions

We characterized the existence of certain steady maps between irreducible sofic shifts by the existence of certain factors onto the minimal right-resolving covers of the image, thus providing characterization of the limit sets of certain stable cellular automata. The most annoying problem is that we do not know if there exist limit sets of stable cellular automata that cannot be reached by (possibly another) cellular automaton with such properties, meaning Conjecture 1 remains a conjecture.

We may note that there exist sofic shifts that have receptive fixed points (and thus are factor of a fullshift by Boyle's lower entropy factors theorem for sofic shifts [**5**, Theorem 3.3]) but whose minimal right-resolving cover does not have a fixed point as depicted on Figure 3: On Figure 3, ${}^{\infty}1^{\infty}$ is a fixed point. It is also a receptive fixed point: 41^*2 is always a valid word and 2 and 4 are magic. Therefore, by Corollary 3.3, this subshift cannot be obtained as the stable limit set of a cellular automaton that is right-continuing almost-everywhere. We do not know if this subshift is a stable limit set of cellular automata:

Question 1. *Is the subshift depicted on Figure 3 a stable limit set of cellular automaton ?*

Note that the minimal left-resolving cover of this sofic shift has a fixed point, this is to keep the example simple; it is left to the reader to modify it so that neither the minimal left nor right resolving covers have a fixed point. By [**15**, Lemma 4.9],

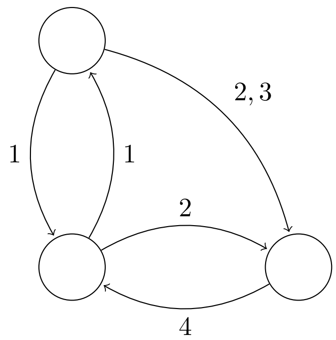

Figure 3. A (non-AFT) mixing sofic shift with a receptive fixed point but whose cover has no fixed point.

an AFT shift which is a factor of a fullshift always has a minimal right-resolving cover with a fixed point. This example shows that it is not the case in general.

We may also note that there exist sofic shifts that have *no* equal entropy SFT factor [**20**]. The example provided in [**20**](Example 2.3) is even worse: It has a receptive fixed point and is such that its minimal right and left-resolving covers both have fixed points, showing that the periodic points obstruction is not the only one. We do not know if [**20**, Example 2.3] can be the stable limit set of a cellular automaton.

Remark that all the stable limit sets of cellular automata constructed in [**15**] and [**3**] have a right-closing *a.e.* dynamics on their limit sets. Therefore, by Corollary 2.3 and Corollary 3.3, these subshifts can be obtained by a right-continuing almost-everywhere and right-closing almost-everywhere cellular automaton. There, obviously, exist stable cellular automata that do not have a right-closing *a.e.* dynamics on their limit set: consider any non right-closing surjective CA. However, it may be possible that its limit set can also be attained by a cellular automaton with such a property:

QUESTION 2. *If* $\mathbf{X}$ *is the stable limit set of a cellular automaton, is it the stable limit set of a cellular automaton which is right-closing almost-everywhere on its limit set? Of a right-continuing* a.e. *cellular automaton ?*

Since a subshift that is weakly conjugate to the stable limit set of a CA is itself the stable limit set of (another) CA [**3**, Lemma 4.1], we may weaken Question 2 to the following:

QUESTION 3. *Is a stable limit set of CA weakly conjugate to a subshift for which such a right-closing or right-continuing almost-everywhere cellular automaton exists ?*

Remark that Question 3 is equivalent to Conjecture 1: If every stable limit set of CA is weakly conjugate to an SFT then since basically any onto endomorphism of an SFT (*e.g.*, the identity which is right-closing *a.e.*) can be obtained as the

dynamics of a stable CA on an SFT limit set [**15**] and can also be attained by a right-continuing factor map. Conversely, by Theorem 2.2 or Theorem 3.2, if there exists such a CA then its limit set is weakly conjugate to an SFT.

A way to construct stable limit sets of CA (and actually, the only method we know) is to prove they are weakly conjugate to an SFT. Moreover, in the constructions, this SFT is always the minimal right-resolving cover of the limit set. We may ask if this is the only way to do it with this technique:

QUESTION 4. *Let $\mathbf{\Lambda}$ be an irreducible sofic shift and $(\mathbf{\Sigma}_R, \pi_R)$ its minimal right-resolving cover. If $\mathbf{\Lambda}$ factors onto $\mathbf{\Sigma}_R$, does it factor onto $\mathbf{\Sigma}_R$ with a right-closing almost-everywhere factor map?*

Remark also that by a theorem of J. Ashley [**1**], two SFTs are weakly conjugate if and only if they are weakly conjugate by right-closing factor maps. We cannot require the same for sofic shifts since a right-closing factor map with SFT range has a SFT domain, but we may ask if it remains true by replacing right-closing by right-closing *a.e.*:

QUESTION 5. *Are two weakly conjugate irreducible sofic shifts also weakly conjugate by right-closing almost-everywhere factor maps?*

Question 4 is a special case of Question 5 since if a sofic shift factors onto its minimal right-resolving cover then they are weakly conjugate.

If the answer to Question 5 is positive, meaning J. Ashley results [**1**] can be extended to sofic shifts, then as a consequence of Boyle's extension lemma [**5**, Lemma 2.4], we can construct a right-closing *a.e.* steady epimorphism of any sofic shift that is weakly conjugate to an SFT and thus by Theorem 2.2 this sofic shift is weakly conjugate to its minimal right-resolving cover. This would mean that we can replace SFT by "the minimal right-resolving cover of the sofic shift" in Conjecture 1. Remark that the answer to Question 5 is positive for AFT shifts as soon as the trivial condition on periodic points is satisfied, for which the proof is short enough to include it here:

PROPOSITION 5.1. *If $\mathbf{X}$ and $\mathbf{Y}$ are two weakly conjugate AFT shifts, with their respective minimal right-resolving covers $(\Sigma_{\mathbf{X}}, \pi_{\mathbf{X}})$ and $(\Sigma_{\mathbf{Y}}, \pi_{\mathbf{Y}})$ such that $Per(\mathbf{X}) \to Per(\Sigma_{\mathbf{Y}})$ and $Per(\mathbf{Y}) \to Per(\Sigma_{\mathbf{X}})$, then there exist right-closing almost-everywhere factor maps from $\mathbf{X}$ onto $\mathbf{Y}$ and from $\mathbf{Y}$ onto $\mathbf{X}$.*

Proof. Let $\varphi : \mathbf{X} \to \mathbf{Y}$ be a factor map from $\mathbf{X}$ onto $\mathbf{Y}$. By [**8**, Theorem 9], $\varphi \circ \pi_{\mathbf{X}}$ can be decomposed through $\pi_{\mathbf{Y}}$ so that $\Sigma_{\mathbf{X}}$ factors $\Sigma_{\mathbf{Y}}$. By the same reasoning, $\Sigma_{\mathbf{Y}}$ factors onto $\Sigma_{\mathbf{X}}$ so that they are weakly isomorphic SFTs; by [**1**, Corollary 1.2], their dimension groups are isomorphic.

Since $\mathrm{Per}(\mathbf{X}) \to \mathrm{Per}(\Sigma_{\mathbf{Y}})$, by [**2**, Theorem 1.2], $\mathbf{X}$ factors onto $\Sigma_{\mathbf{Y}}$ by a right-closing *a.e.* factor map, and thus also factors onto $\mathbf{Y}$ by a right-closing *a.e.* factor map. By the same reasoning, $\mathbf{Y}$ factors onto $\mathbf{X}$ by a right-closing *a.e.* factor map. □

Remark that by [**15**, Lemma 4.9] the condition on periodic points hold if the AFT shifts have a receptive fixed point and therefore the answer to Question 5 is positive for AFT shifts with a receptive fixed point.

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